Prof. Dale van Harlingen, UIUC, Physics 498 Superconducting Quantum Devices

Ginzburg-Landau theory --- supercurrents and fluxoid quantization

Discussion the Ginzburg-Landau theory in three parts:

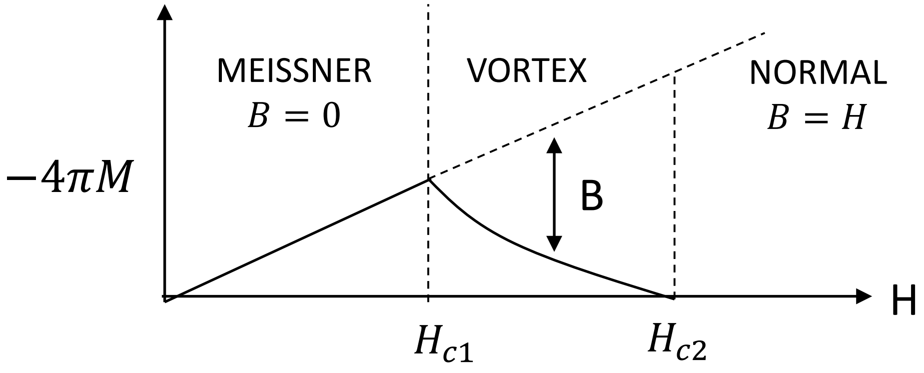
1. Presentation of the model and derivation of the penetration length and coherence length
2. Calculation of the surface energy and categorization of Type I and Type II superconductivity
3. Current-carrying states and phase coherence

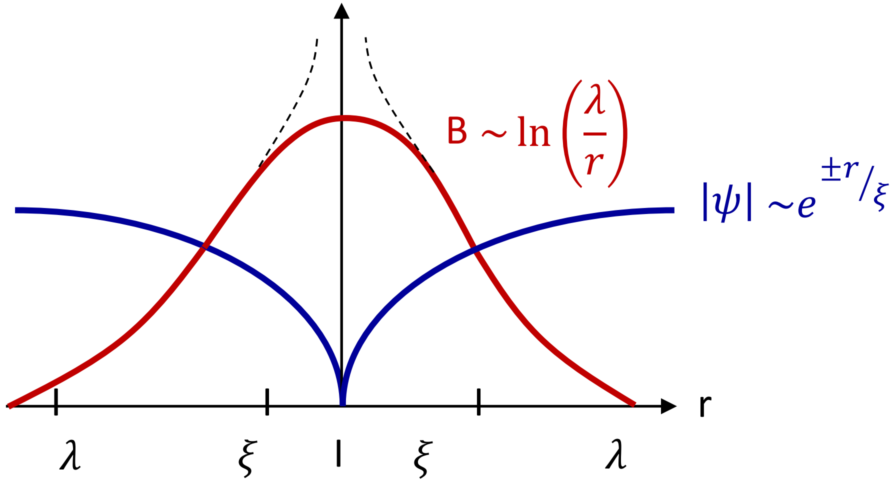
Type II Superconductivity $H > "H_c" \Longrightarrow$ field penetrates in discrete vortices

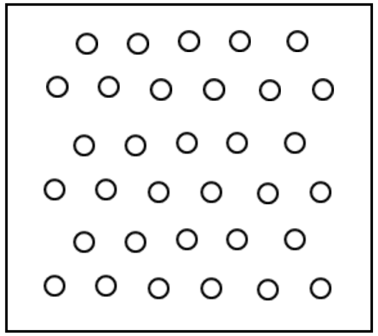
VORTEX STATE

(1) $\kappa = \frac{\lambda}{\xi} > \frac{1}{\sqrt{2}}$ $\Longrightarrow$ negative surface energy $\Longrightarrow$ maximize N-S interface area

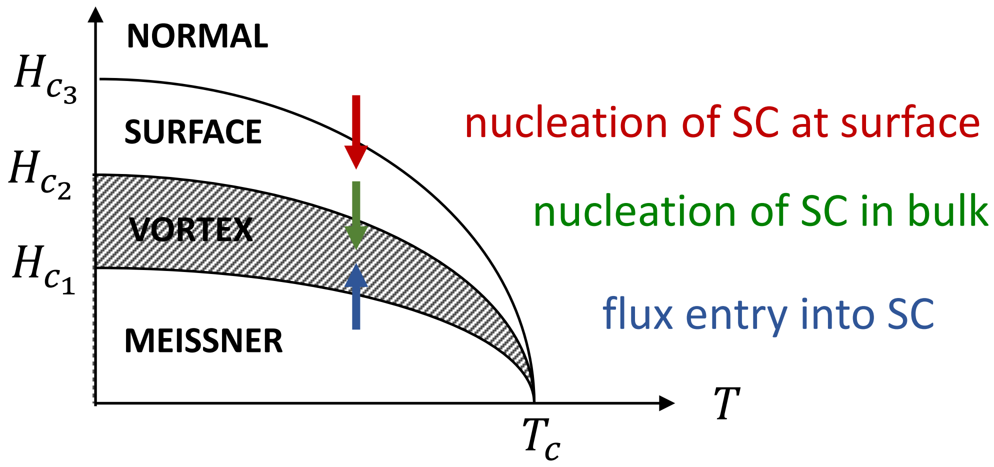
(2) Fluxoid quantization $\Longrightarrow$ smallest flux unit $= \Phi_o$

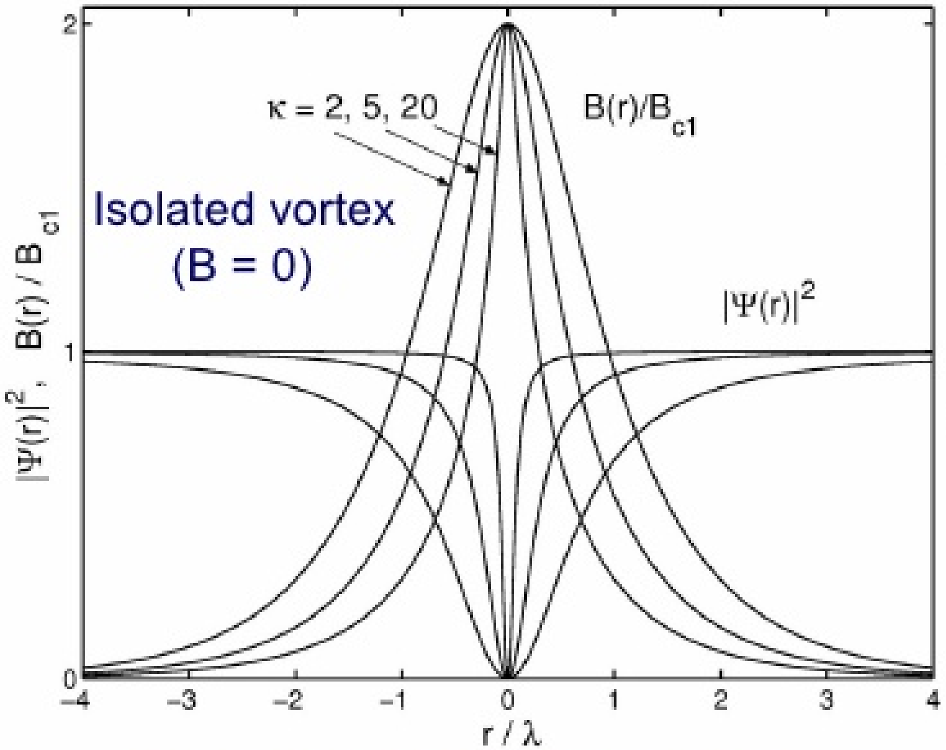


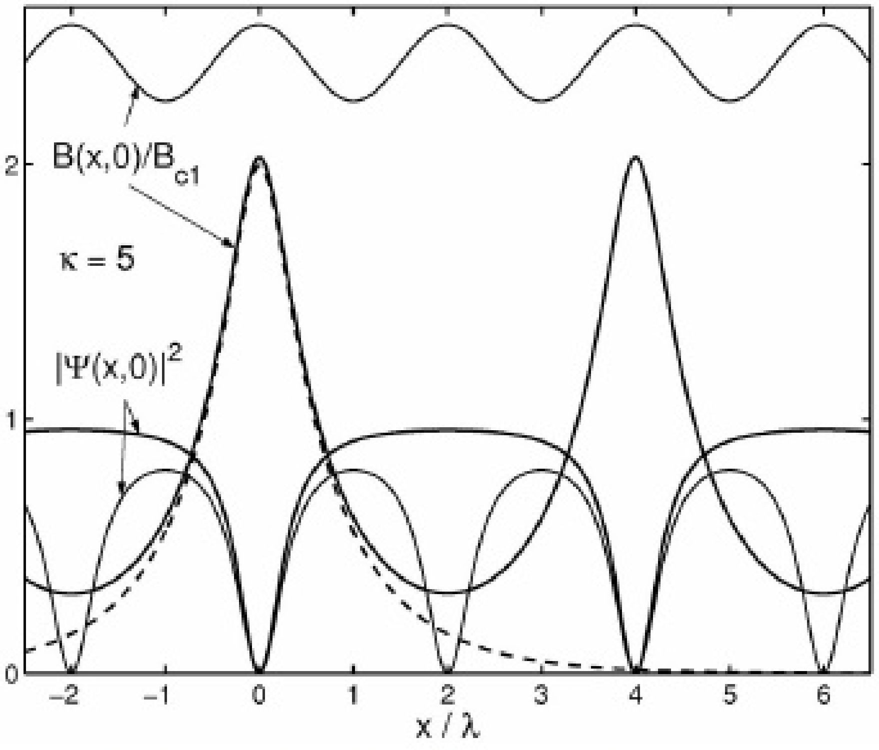




Vortex density: $n = \frac{B}{\Phi_o}$







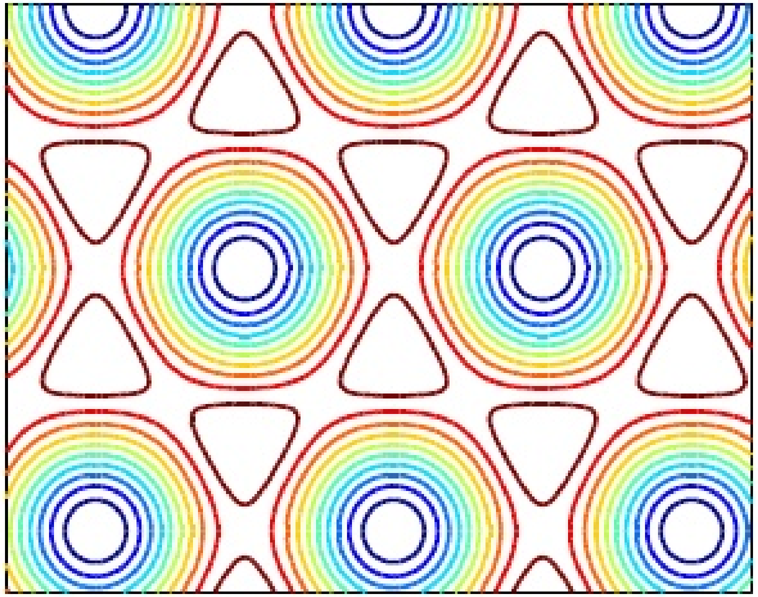
Vortex lattice: B = B_0 and $4B_0$

vortex spacing: a = 4λ and 2λ

Bulk superconductor

Ginzburg-Landau theory

EHB, PRL 78, 2208 (1997)



← Abrikosov solution near B_{c2}:
stream lines = contours of |ψ|2 and B

When do these states form?

Nucleation of SC in the bulk

$\Rightarrow$ H_{c2} "upper critical field"

When does SC start as H is lowered?

$$H_{c2} = \frac{\Phi_0}{2_\pi \xi^2} = \frac{4_\pi \lambda^2 H_c^2}{\Phi_0} = \sqrt{2}\,\kappa\, H_c$$ "upper critical field"

Vortex Nucleation

$\Rightarrow$ H_{c1} "lower critical field"

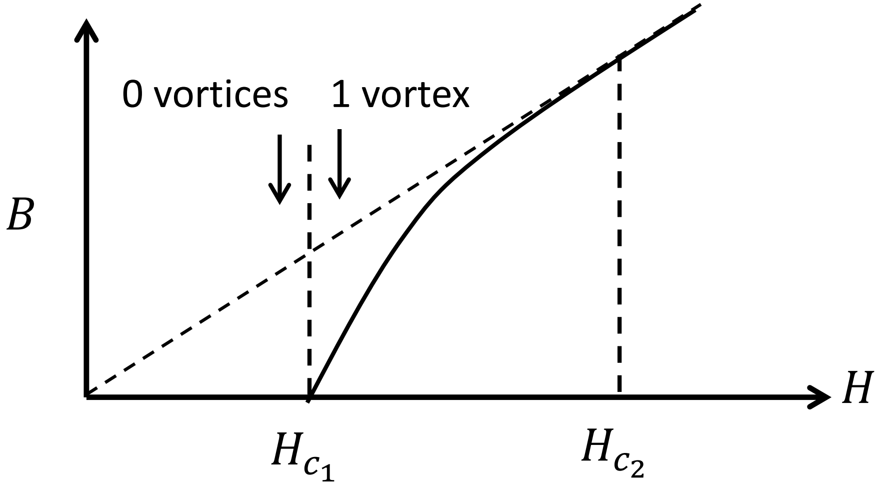
When do vortices enter as H is raised?

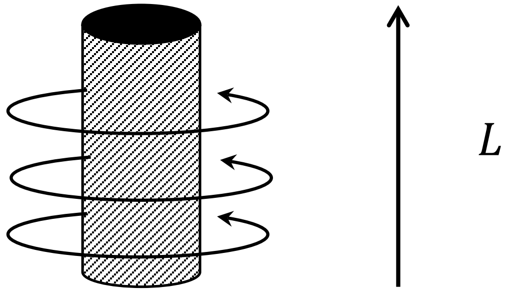
Vortex Nucleation:

When do vortices enter as H is raised?

Define critical field H_{c_1}

At H_{c_1} $\quad G_S^{Meissner} = G_s^{vortex}$





Let $\varepsilon_\ell =$ line energy of vortex/length

Tradeoff vortex energy vs. field energy (to allow B to penetrate when applied field is H_{c_1})

$$\varepsilon_\ell L = \frac{1}{4_\pi}\int \vec{B}\cdot\vec{H}\,dV \quad = \frac{1}{4_\pi}\int H_{c_1} B\,dV = \frac{H_{c_1}}{4_\pi}\left(\int B\,dA\right)L \quad = \frac{H_{c_1}}{4_\pi}\,\Phi_0\,L$$

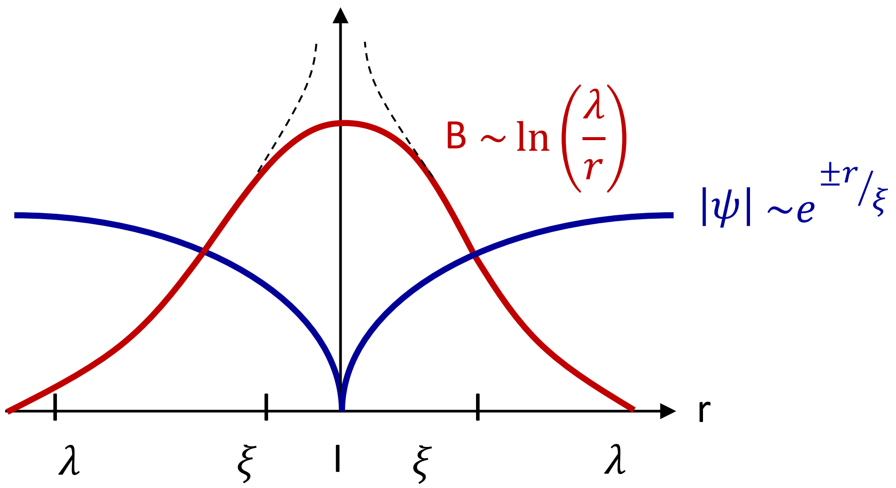
$$\therefore H_{c_1} = \frac{4_\pi \varepsilon_\ell}{\Phi_0}$$ "lower critical field"

ε_ℓ? Must solve GL to get vortex slope: $\psi(r), A(r)$

Calculate line energy (field energy $+KE$

Guess: $\varepsilon_\ell \sim \left(\frac{H_c^2}{8_\pi}\right)\lambda^2 - \left(\frac{H_c^2}{8_\pi}\right)\xi^2 \sim \left(\frac{H_c}{8_\pi}\right)^2 (\lambda^2 - \xi^2)$

field energy — condensation energy



Solutions $\left(\kappa \gg \frac{1}{\sqrt{2}}\right)$ use full GL

$\psi(r) \sim |\psi_\infty| \tanh \frac{r}{\xi}$

$$B(r) = \left(\frac{\Phi_0}{2_\pi\lambda^2}\right) K_0\left(\frac{r}{\lambda}\right) = H_{c_2} K_0\left(\frac{r}{\lambda}\right) = \begin{cases} \frac{\Phi_0}{2_\pi\lambda^2}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\lambda}{r}\right)^{1/2} e^{-r/\lambda} & r \gg \lambda \quad \text{(long range)} \\ \frac{\Phi_0}{2_\pi\lambda^2}\left[\ell n\left(\frac{\lambda}{r} + 0.12\right)\right] & \xi \ll r \ll \lambda \quad \text{(short range)} \end{cases}$$

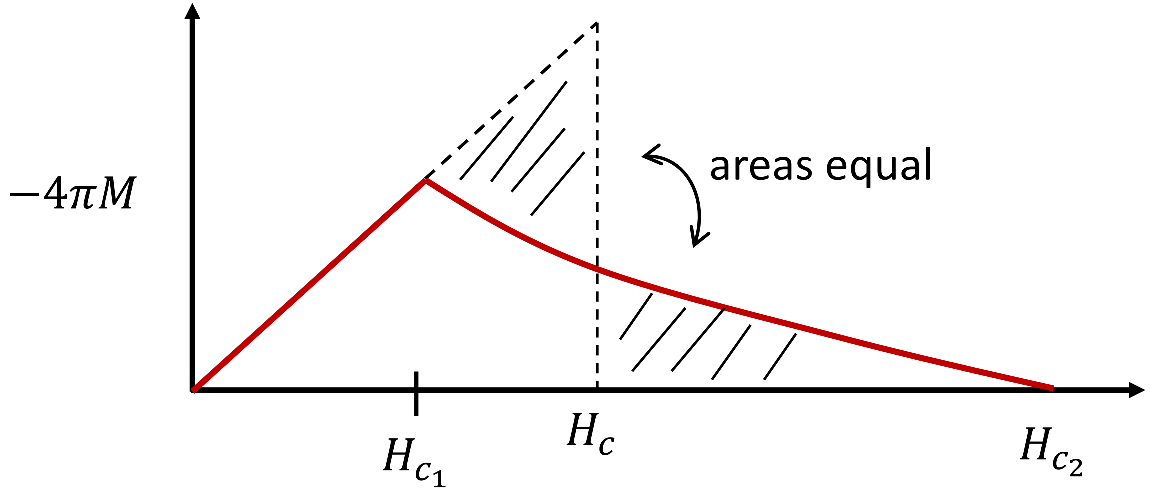
zero – order Hankel function

$B(r)$ does not diverge in core – flattens off as $|\psi|^2 \to 0$ near center

Find for the line energy: $\varepsilon_\ell = \left(\frac{\Phi_0}{4_\pi \lambda}\right)^2 \ell n\,\kappa = \left(\frac{H_c^2}{8_\pi}\right) 4_\pi \xi^2 \ell n\,\kappa$

$$H_{c_1} = \frac{4_\pi}{\Phi_0}\varepsilon_1 = \frac{\Phi_0}{4_\pi \lambda^2}\ell n\,\kappa = H_c \frac{\ell n\,\kappa}{\sqrt{2}\,\kappa}$$

$$\left.\begin{aligned} H_{c_1} &= H_c \frac{\ell n\,\kappa}{\sqrt{2}\,\kappa} \\ H_{c_2} &= \sqrt{2}\,\kappa\, H_c \end{aligned}\right\} \quad H_c = \frac{1}{\sqrt{\ell n\,\kappa}}\left(H_{c_1} H_{c_2}\right)^{1/2}$$



$$H_{c_1} \sim \frac{\Phi_0}{\lambda^2}$$

$$H_{c_2} \sim \frac{\Phi_0}{\xi^2}$$

Current-Carrying Situations

$GL2$: cannot neglect phase --- ψ is complex

$$\bar{J}_s = \frac{e^*\hbar}{2m^* i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{(e^*)^2}{m^* c}\psi^*\psi\vec{A}$$

Let $\psi = |\psi| e^{i\theta} = ({n_s}^*)^{1/2} e^{i\theta}$

$$\vec{J}_s = \frac{e^*\hbar}{m^*} n_s^* \vec{\nabla}\theta - \frac{(e^*)^2}{m^* c} n_s^* \vec{A} \quad = n_s^* \frac{e^*\hbar}{m^*}\left(\vec{\nabla}\theta - \frac{e^*}{\hbar c}\vec{A}\right)$$

alternate descriptions

$$\vec{J}_s = n_s^* \frac{e^*\hbar}{m^*}\vec{\nabla}\phi$$

$$= n_s \frac{e\hbar}{2m}\vec{\nabla}\phi$$

$$\phi = \theta - \frac{2e}{\hbar c}\int \vec{A}\cdot\overrightarrow{d\ell}$$

“gauge-invariant phase”

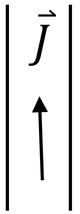
$$\vec{J}_s = n_s^* e^* \vec{v}_s$$

$$= n_s e \vec{v}_s$$

$$\vec{p} = \frac{\hbar}{2m}\vec{\nabla}\theta = m^*\vec{v}_s + \frac{e^*}{c}\vec{A}$$

“canonical momentum”

<u>Critical Current</u> wire or film ($d \ll \xi, \lambda$)

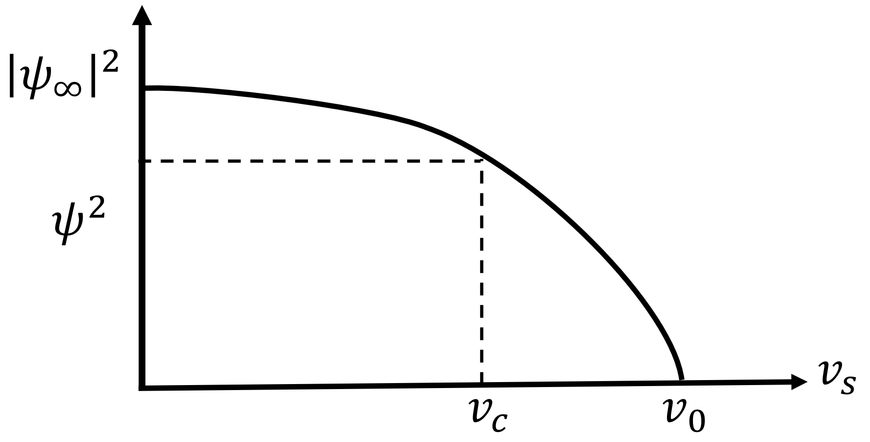


$|\psi|$ uniform but may be suppressed $< |\psi_\infty|$ $\vec{\nabla}|\psi| = 0$ $H = B$ (non-screening)

$$\vec{J}_s = |\psi|^2 e v_s^* \le n_s^* e^* v_s^*$$

$$\Delta G = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{\hbar^2}{2m^*}\left(\vec{\nabla}|\psi|\right)^2 + \left(\frac{1}{2}m^* v_s^2\right)|\psi|^2 + \frac{1}{8\pi}(H - B)^2 \quad = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2}m^* v_s^2|\psi|^2$$

Minimize ΔG wrt $|\psi|^2 \Rightarrow$ $\alpha + \beta|\psi|^2 + \frac{1}{2}m^* v_s^2 = 0$



$$|\psi|^2 = \frac{\alpha}{\beta} - \frac{1}{2}\frac{m^* v_s^2}{\beta}$$

$$= |\psi_\infty|^2\left(1 - \frac{1}{2}\ \frac{m^* v_s^2}{|\alpha|}\right) \qquad |\psi_\infty|^2 = -\frac{\alpha}{\beta}$$

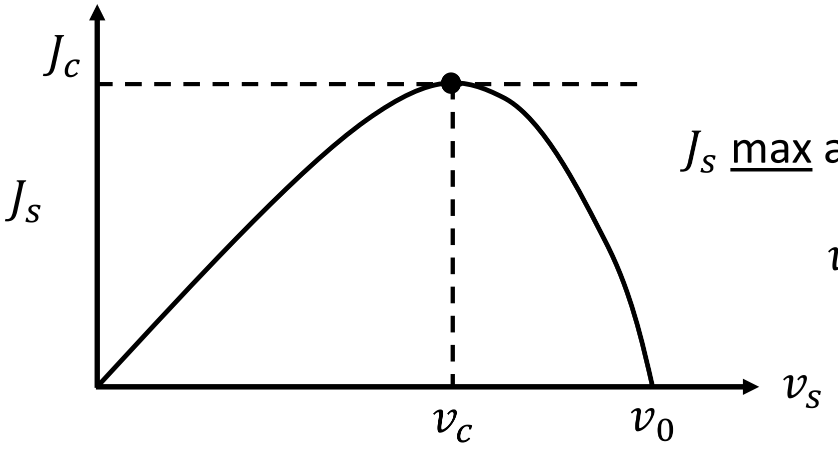
$$= |\psi_\infty|^2\left[1 - \left(\frac{\xi m^* v_s}{\hbar}\right)^2\right] \qquad \xi = \left(\frac{\hbar^2}{2m^*|\alpha|}\right)^{1/2}$$

$$= |\psi_\infty|^2\left[1 - \left(\frac{v_s}{v_0}\right)^2\right] \qquad v_0 = \frac{\hbar}{\xi m^*}$$

$|\psi|^2 \to 0$ at $v_0 = \frac{\hbar}{\xi m^*}$

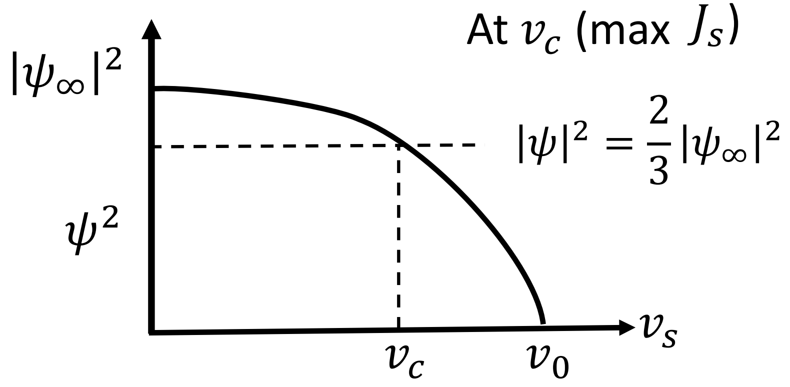
$\xi = \frac{\hbar}{m^* v_0}$ Like a deBroglie wavelength

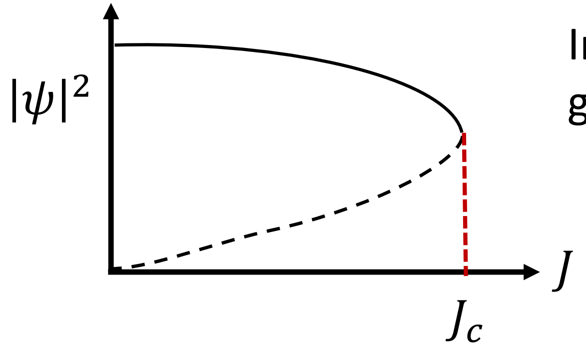
$$\vec{J}_s(v_s) = |\psi|^2 e^* v_s = |\psi_\infty|^* e^* v_s \left[1 - \left(1 - \frac{\xi m^* v_s}{\hbar}\right)^2\right]$$



J_s max at $v_c = \frac{1}{\sqrt{3}} \frac{\hbar}{\xi m^*} = \frac{1}{\sqrt{3}} v_0$

$v_c \sim 0.6 v_0$





Increase current, wire goes normal at J_c

$$J_c = \frac{2}{3\sqrt{3}} \frac{e^*\hbar}{m^*\xi} |\psi_\infty|^2$$ "GL depairing current"

$$= \frac{c}{3\sqrt{6}\pi} \frac{H_c(T)}{\lambda(T)} \sim \frac{1-t}{(1-t)^{-1/2}} \sim (1-t)^{3/2}$$

This is observed in experiments if piling up of currents can be avoided by using small wires or ground-planed thin films

Compare to London model: $|\psi| = |\psi_\infty|$ always

$$\left(\frac{1}{2} m^* v_c^2\right) n_s^* = \frac{H_c^2}{8\pi} \Rightarrow v_c = \left(\frac{H_c^2}{4\pi n_s^* m^*}\right)^{1/2}$$

$$J_c^L = n_s^* e^* v_s = \left(\frac{H_c^2 n_s^* (e^*)^2}{4\pi m^*}\right)^{1/2} = \frac{c}{4\pi} \frac{H_c}{\lambda} = \frac{\sqrt[3]{6}}{4} J_c^{GL}$$

Larger (x 1.84) -- neglects suppression of $|\psi|$ by the current

Flux Quantization

$GL \rightarrow$ microscopic coherence r ξ (range of changes in ψ)

microscopic phase coherence over long range (∞ ?)

$$\vec{J}_s = \frac{e^*\hbar}{2m^*i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{(e^*)^2}{m^*c}(\psi^*\psi)\vec{A}$$

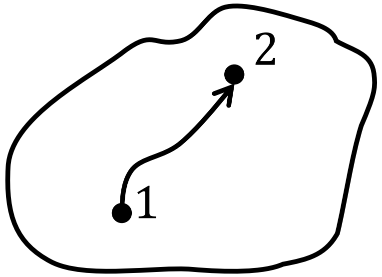
$$\psi = (n_s^*)^{\frac{1}{2}}\, e^{i\theta}$$

$$\vec{J}_s = \frac{e^*\hbar}{m^*} n_s^* \vec{\nabla}\theta - \frac{(e^*)^2}{m^*c} n_s^* \vec{A}$$

$$= n_s^* \frac{e^*\hbar}{m^*} \underbrace{\left(\vec{\nabla}\theta - \frac{e^*}{\hbar c}\vec{A}\right)}_{\vec{\nabla}\phi}$$

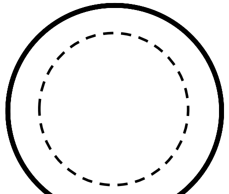
$$\phi = \theta - \frac{e^*}{\hbar c}\int_1^2 \vec{A}\cdot\overrightarrow{d\ell}$$

= gauge invariant phase



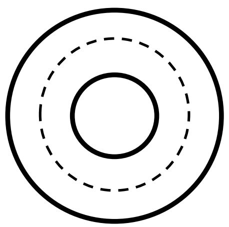
ψ needs to single-valued which forces a phase constraint:

Singly-connected



$$\oint \vec{\nabla}\theta\cdot\overrightarrow{d\ell} = 0$$

Multiply-connected



$$\oint \vec{\nabla}\theta\cdot\overrightarrow{d\ell} = 2\pi n$$

$$\frac{\hbar c}{e^*}\vec{\nabla}\theta = \vec{A} + \frac{m^*c}{n_s(e^*)^2}\vec{J}_s$$

$$\frac{\hbar c}{e^*}\oint \vec{\nabla}\theta\cdot\overrightarrow{d\ell} = \oint \vec{A}\cdot\overrightarrow{d\ell} + \frac{m^*c}{n_s(e^*)^2}\oint \vec{J}_s\cdot\overrightarrow{d\ell}$$

$$\frac{\hbar c}{e^*}(2\pi n) = \Phi + \frac{m^*c}{n_s^*(e^*)^2}\oint \vec{J}_s\cdot\overrightarrow{d\ell}$$

"fluxoid" | magnetic flux | kinetic flux

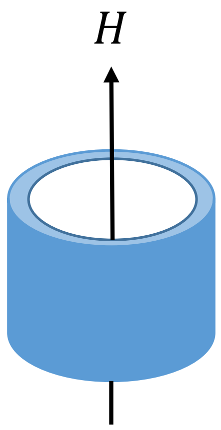
Fluxoid quantized in units of $\Phi_0 = \frac{hc}{2e}$ "flux quantum" $\Phi_0 = 2.07 \times 10^{-15}\, Wb = 2.07 \times 10^{-7}\ G - cm^2$

Fluxoid $\quad \Phi' = \frac{\Phi_0}{2\pi} \oint \vec{\nabla}\theta \cdot \overrightarrow{d\ell} = n\,\Phi_0$

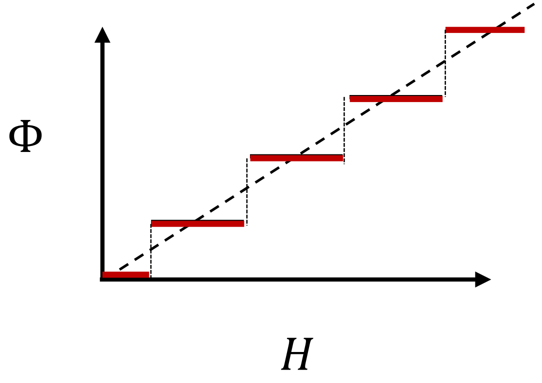
Special case : Bulk $SC \rightarrow \vec{J}_s = 0$

$\Phi' = \Phi = n\Phi_0$

Deaver & Fairbank (1961)



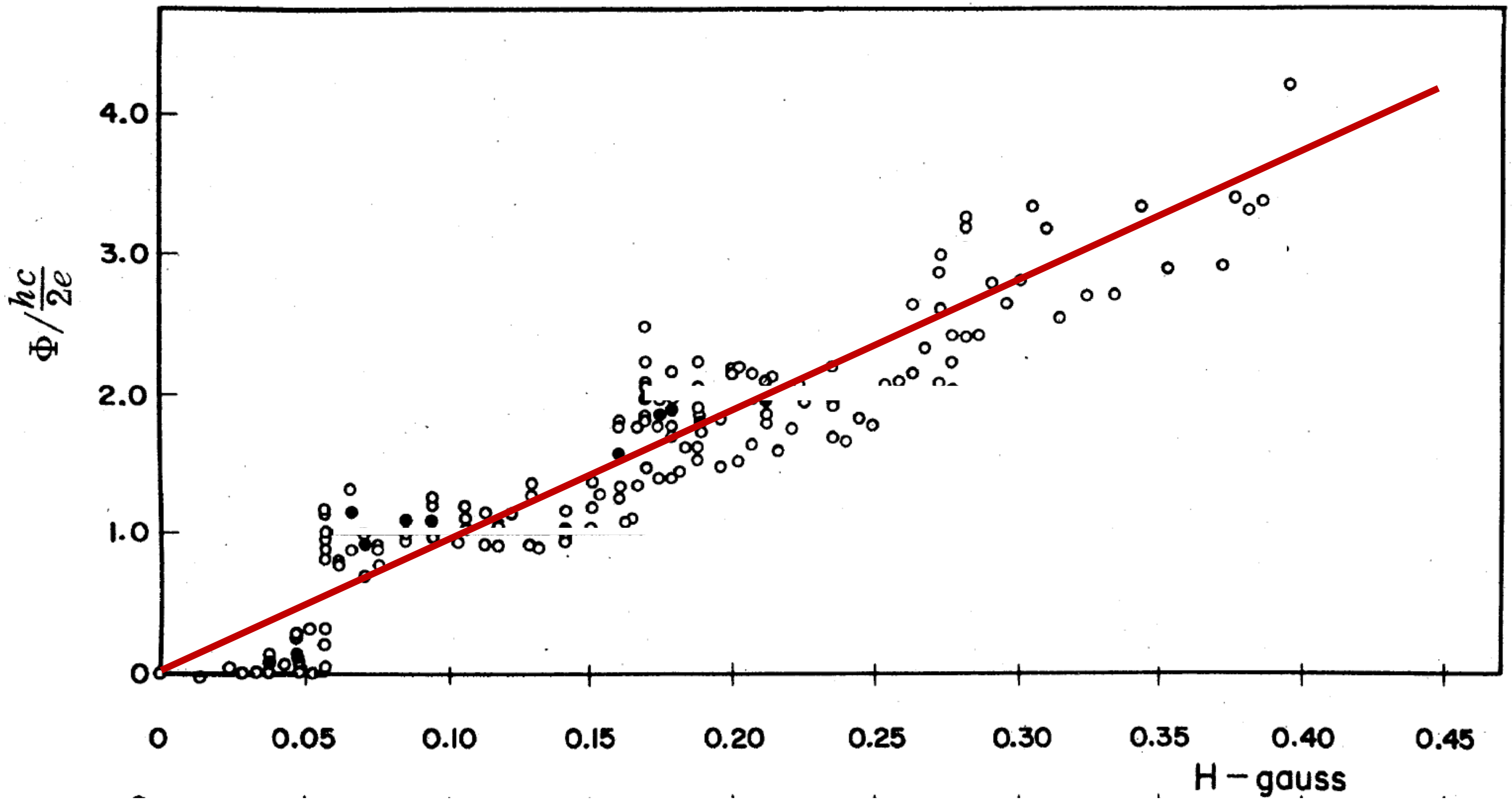
field-cooled
superconductor
cylinder



$e^* = 2e!$

EXPERIMENTAL EVIDENCE FOR QUANTIZED FLUX IN SUPERCONDUCTING CYLINDERS*

Bascom S. Deaver, Jr., and William M. Fairbank
Department of Physics, Stanford University, Stanford, California
(Received June 16, 1961)

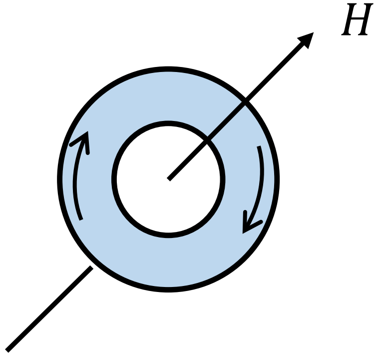


How can flux be quantized if $\vec{J}_s = 0$?

★ CURRENTS flow within λ of surfaces

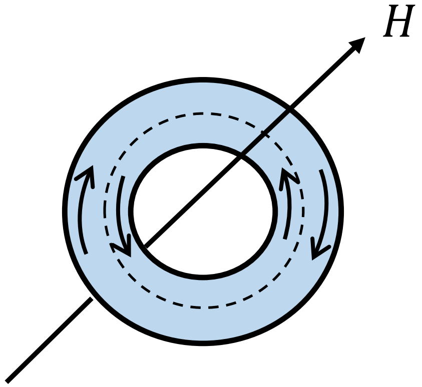
<u>Zero-field cooled</u>

$\Phi = 0 \quad (n = 0)$



Surface currents flow on OUTSIDE as field is applied – screens bulk of SC and hole

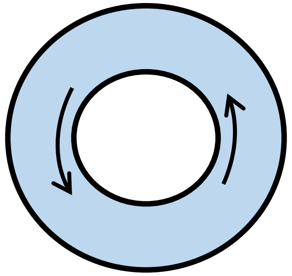
<u>Finite-field cooled</u>



Surface currents flow on INSIDE to maintain flux in hole, and on OUTSIDE to screen bulk

(some flux is pushed in, some out from bulk area to make Φ quantized inside)

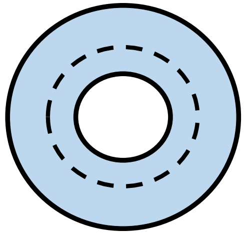
$H \to 0$



INSIDE currents persist
OUTSIDE currents reduce

$\Longrightarrow$ TRAPPED FLUX

Fluxoid Quantization:



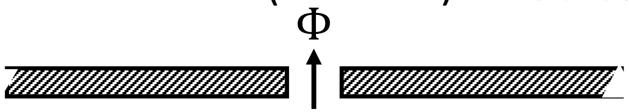
$$\Phi' = \Phi + \frac{mc}{n_s e^2}\oint \vec{J}_s \cdot \overrightarrow{d\ell} = n\Phi_0 = \Phi + \frac{mc}{e}\oint \vec{v}_s \cdot \overrightarrow{d\ell}$$

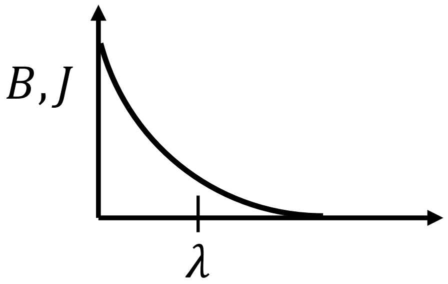
$\vec{J}_s = 0 \Rightarrow \Phi = n\Phi_0$ flux quantization

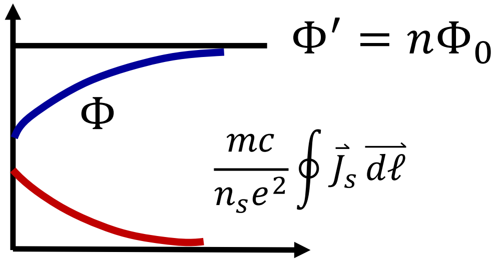
$\vec{J}_s \neq 0 \Rightarrow \Phi' = n\Phi_0$ flux is not quantized

SITUATIONS where this applies

(1) Near surfaces (within λ) SC sample with trapped flux







(2) Near vortex core (within λ)

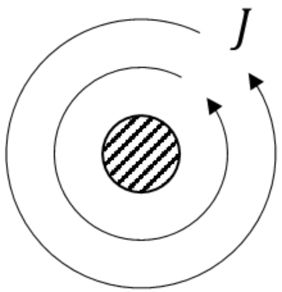
(3) Thin samples $(w < \lambda) \Rightarrow$ Little-Parks experiment

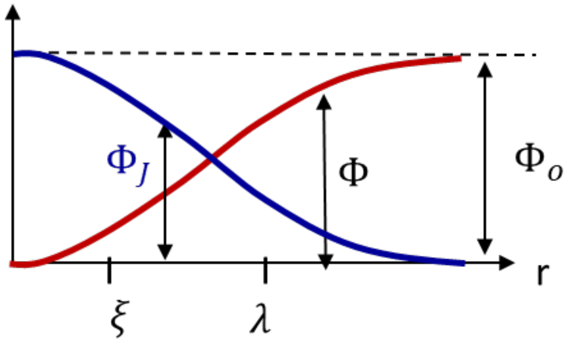
(4) Rotating sample $\Rightarrow$ London rotation

(5) Transport current $\vec{J}_n \Rightarrow$ thermoelectric effect

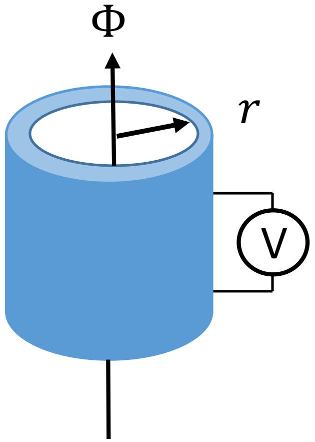
(6) SC weak links $\Rightarrow$ Josephson effect

(7) Unconventional $SC \Rightarrow d$-wave, p-wave symmetry



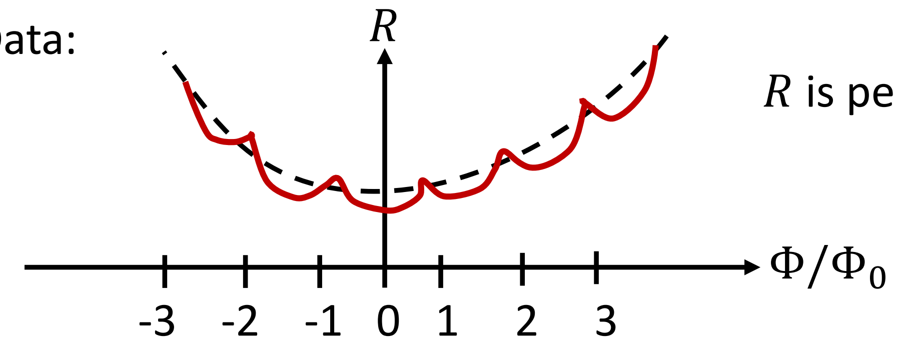


(3) Little-Parks Experiment (1962)



Measure resistance of a superconductor cylinder just above T_c

Data:



R is periodic in Φ

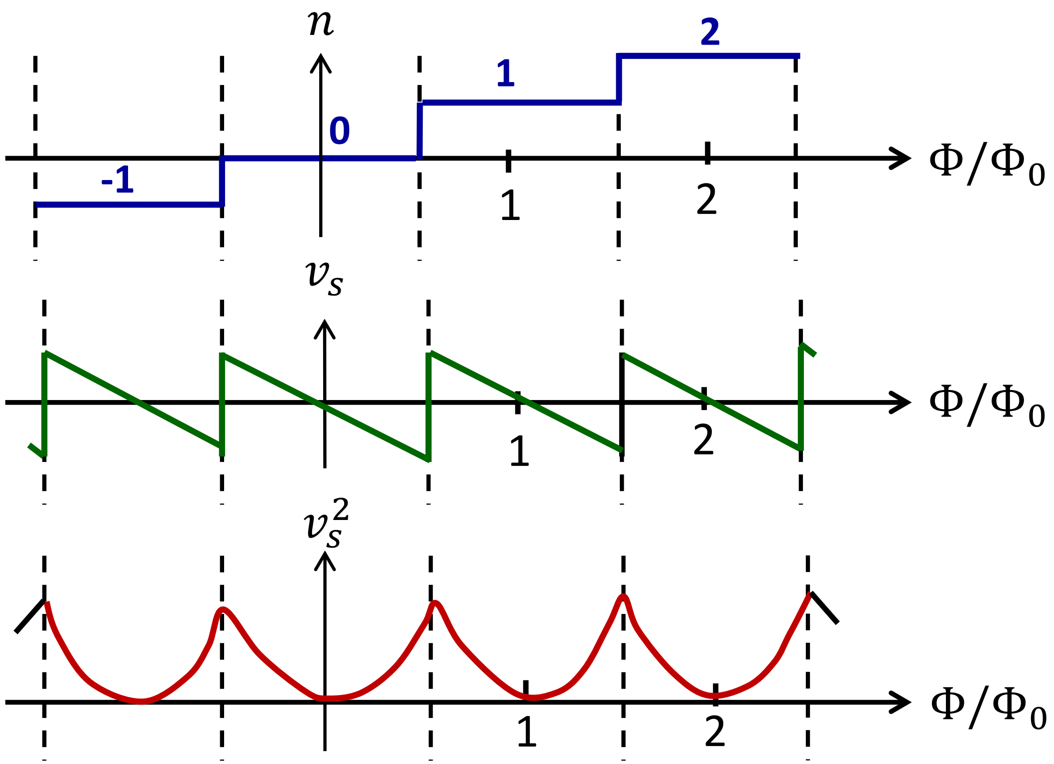
Model:

$$\Phi' = n\Phi_0 = \Phi + \frac{mc}{e}\oint \vec{v}_s\,\overrightarrow{d\ell} = \Phi + \frac{mc}{c}(2\pi r)v_s$$

$$v_s = \frac{\hbar}{mr}\left(n - \frac{\Phi}{\Phi_0}\right)$$

v_s determined by n and Φ --- maximize ΔG $\Rightarrow$

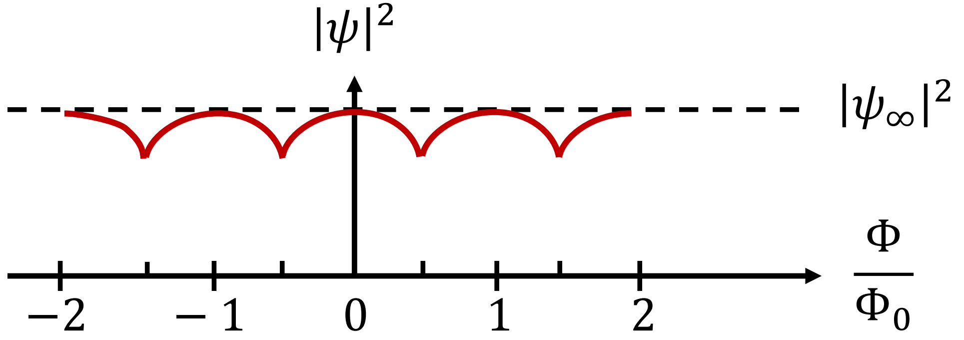
$|v_s|$ small as possible: $\Delta G \sim n_s\left(\frac{1}{2}m^* v_s^2\right)$



Recall variation of $|\psi|$ with v_s:

$$|\psi|^2 = |\psi_\infty|^2 \left[1 - \left(\frac{\xi m^* v_s}{\hbar}\right)^2\right]$$

$$= |\psi_\infty|^2 \left[1 - \left(\frac{2\xi}{r}\right)^2 \left(n - \frac{\Phi}{\Phi_0}\right)^2\right]$$



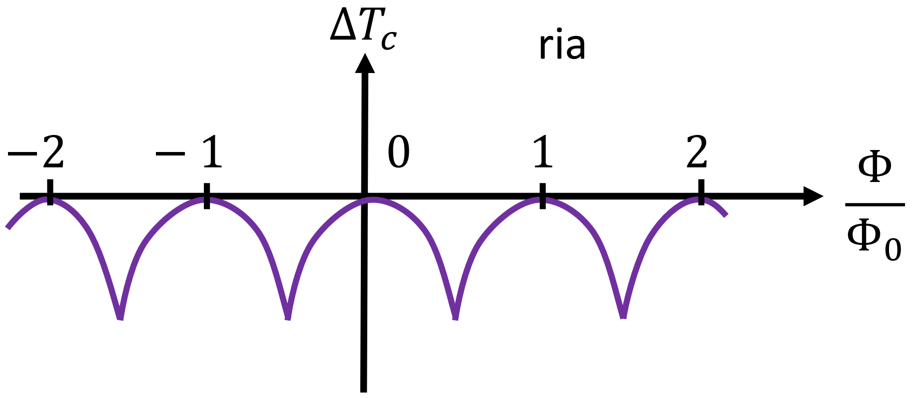
Transition temperature T_c when $|\psi|^2 \to 0 \quad (d \ll \xi, \lambda)$

$$\frac{1}{\xi^2} = \left(\frac{2}{r}\right)^2 \left(n - \frac{\Phi}{\Phi_0}\right)^2 \sim \frac{1-t}{\xi_0^2}$$

$$\xi \sim \frac{\xi_0}{(1-t)^{1/2}}$$

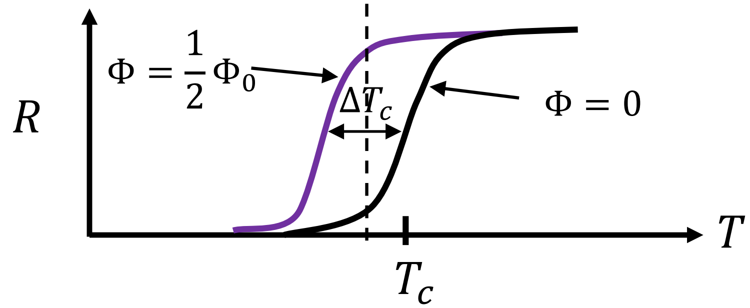
$$1 - t = \frac{\Delta T_c}{T_c} \sim \left(\frac{2\xi_0}{r}\right)^2 \left(n - \frac{\Phi}{\Phi_0}\right)^2$$

$$t = \frac{T}{T_c}$$



Max suppression is $\frac{\Delta T_c}{T_c} \sim \left(\frac{\xi_0}{r}\right)^2$ at $\Phi = \left(n + \frac{1}{2}\right)\Phi_0$

Shows up experimentally as a variation in R (at constant T):

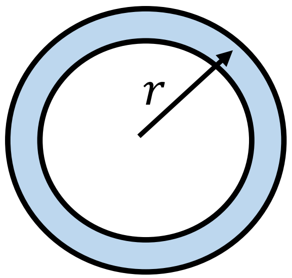


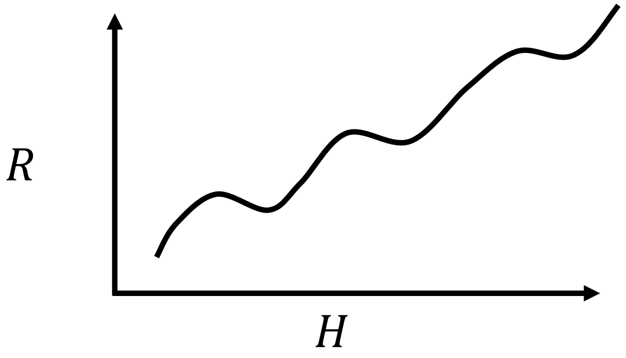
Significance of Little-Parks experiments:

(1) showed reality of the "fluxoid"

(2) demonstrated use of GL free energy to understand experiments

Sharvin & Sharvin repeated this experiment for nanoscale <u>normal</u> rings:





Observed quantized magnetoresistance oscillations for

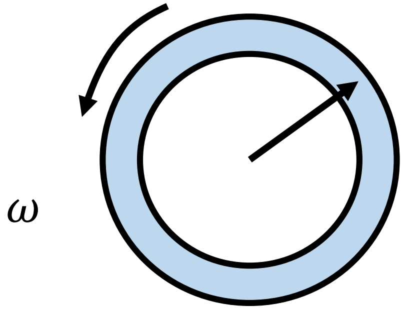
$$2\pi r < \ell_\phi = v_F \tau_\phi$$

phase coherence length in N $(< 1\mu m)$

This is due to phase coherence in normal metals over microscopic scales, <u>not</u> superconductivity

Important result in nanoscale physics

(4) London rotation – spinning SC ring (thin)



In rest frame, $\oint \vec{J}_s \cdot \overrightarrow{d\ell} \neq 0$

$$\Phi = \overset{0}{\cancel{n}}\Phi_0 - \frac{mc}{e}\oint \vec{v}_s \cdot \overrightarrow{d\ell}$$

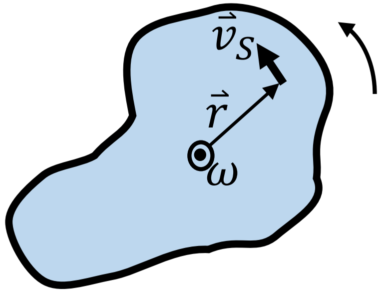
$$= -\frac{mc}{e}(\omega R)2\pi R = -\frac{2mc}{e}(\pi R^2)\omega = -\frac{2mc}{e}A\omega$$

Net magnetic flux generated due to rotation:

"London moment"

$A =$ Area of loop

General Rotation:



$$\vec{v}_s = \vec{\omega} \times \vec{r}$$

$$\Phi = -\frac{mc}{e}\oint \vec{v}_s \cdot d\ell = -\frac{mc}{e}\int_A (\vec{\nabla} \times \vec{v}_s) \cdot \overrightarrow{dA} = -\frac{mc}{e}\int_A 2\vec{\omega} \cdot \overrightarrow{dA}$$

using $\vec{\nabla} \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega}$ from classical mechanics

$$\Phi = -\frac{2mc}{e}A\,\omega$$

$$\vec{B} = -\frac{2mc}{e}\,\vec{\omega}$$ uniform magnetic field

How big? $B = \frac{2m}{e}\omega \quad (mks)$

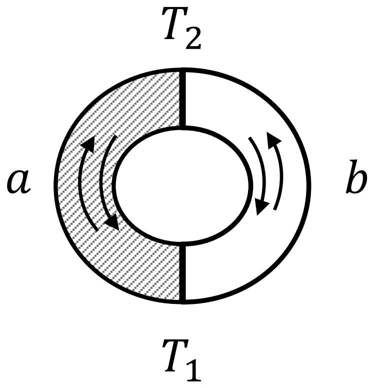
$= 7 \times 10^{-11}\, T/H_z$

$\Phi = 3.5\,(\Phi_0/cm^2)/H_z$

Can be used to detect rotations. Does not require a hole.

(5) Transport   If $\vec{J}_n \neq 0$, then $\vec{J}_s \neq 0$ since $\vec{J} = 0$ so that $\vec{B} = 0$

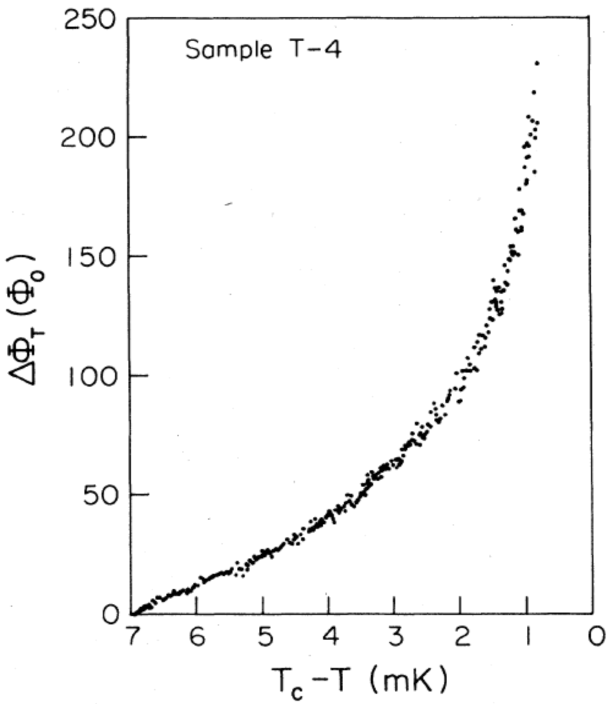
Example: thermoelectric effects in superconductors



$$\vec{J}_n = L_T \vec{\nabla} T = -\vec{J}_s$$

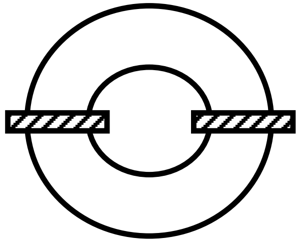
$$\Phi = n\Phi_0 + \frac{mc}{e^2}\int_{T_1}^{T_2} dT \left(\frac{L_T^a}{n_s^a} - \frac{L_T^b}{n_s^b}\right)$$

Fluxoid, not flux, is quantized!



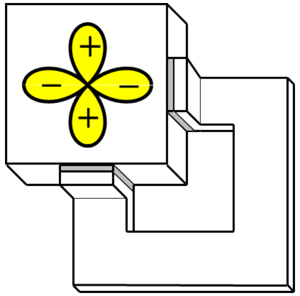
Can also do with ultrasound (phonon drag to get $\vec{J}_n$)

(6) Josephson Effect and SQUIDs



SC interrupted with one or more weak link sections with discrete $\nabla\phi$

(7) Unconventional *SC*



SC interrupted with a superconductor with phase anisotropy, giving a discrete phase change between different tunneling drirections

d-wave, p-wave, etc.